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Approximating soft-capacitated facility location problem with uncertainty

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Abstract In this paper we devise the stochastic and robust approaches to study the soft-capacitated facility location problem with uncertainty. We first present a new stochastic soft-capacitated model called *The 2-Stage Soft Capacitated Facility Location Problem* and solve it via an approximation algorithm by reducing it to linear-cost version of 2-stage facility location problem and dynamic facility location problem. We then present a novel robust model of soft-capacitated facility location, *The Robust Soft Capacitated Facility Location Problem*. To solve it, we improve the approximation algorithm proposed by Byrka et al. (LP-rounding algorithms for facility-location problems. CoRR, 2010a) for RFTFL and then treat it similarly as in the stochastic case. The improvement results in an approximation factor of $\alpha + 4$ for the robust fault-tolerant facility location problem, which is best so far.

Keywords Facility location · Approximation algorithm · 2-Stage · Robust · Stochastic · Soft-capacitated

1 Introduction

Given two discrete sets of clients and candidate locations where to build facilities, in the classic *Uncapacitated Facility Location Problem* (UFL) we are to pick some

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locations and build facilities on them so that each client can connect itself to a facility, in order to minimize the building and connection cost. There are a lot of variations of this problem (e.g. Li et al. 2012b). Such models capture some essence of facility location, but fail to deal with the uncertainty in decision making that is very common in realistic settings.

Usually facility location decisions are expensive and hard to reverse, and their influence lasts a long time. Any of the parameters of the problems, e.g. costs, demands, distances, may change greatly during the time when design decisions are in effect. Further, because of poor measurements or tasks inherent in the modeling process such as aggregating demands points and choosing a distance norm, parameter estimates may also be inaccurate. For the sake of this, researchers have been developing models for facility location under uncertainty for several decades (cf. Snyder 2007).

One approach to handle the uncertainty is the study of *The 2-Stage Stochastic Facility Location Problem* (TSFL) (see Byrka et al. 2010a; Srinivasan 2007; Swamy and Shmoys 2005; Swamy 2004). In which, there are two stages of decision making and a set of scenarios which are subsets of the set of all clients, each happening with a prefixed probability. Upon the first stage, we open facilities without the specific knowledge of clients; then in the second stage the realized scenario enters, we are allowed to open additional facilities but with much higher opening cost, and connect the clients to the opened facilities. The aim is to minimize the facility opening cost of the first stage plus the expected cost incurred in the second stage. This problem is introduced by Swamy and Shmoys (2005). The approximation ratio is then improved by Srinivasan (2007) to 2.369. Byrka et al. (2010a) use the techniques they developed for UFL to obtain a 2.298-approximation algorithm. The best ratio is 1.86, due to Ye and Zhang (2006). In that paper they solve *The Dynamic Facility Location Problem* (DFL) using primal-dual algorithm combined with greedy procedures, and TSFL is just a special case of DFL. There is another trend of analysis that derives *per-scenario* bound for TSFL. Swamy (2004) obtains a 3.378 bound and is improved by Srinivasan (2007) to 3.25. Then Byrka et al. (2010a) further improves the bound to 2.496. (However, they have overlooked best parameters and their algorithm is in fact of ratio 2.425.)

The Soft-Capacitated Facility Location Problem (SCFL) is similar to the UFL except there is a soft-capacity u_i associated with each facility i, which means if we want this facility to serve k clients, we have to open it $\lceil k/u_i \rceil$ times at a cost of $\lceil k/u_i \rceil f_i$. This problem is also known as *facility location problem with integer decision variables* in the operations research literature (see Bauer and Enders 1997). Chudak and Shmoys (1999) gave a 3-approximation algorithm for SCFL with uniform capacities using LP rounding. For non-uniform capacities, Jain and Vazirani (2001) showed how to reduce this problem to the UFL, and by solving the UFL through a primal-dual algorithm, they obtained a 4-algorithm. A local search algorithm proposed by Arya et al. (2001) had an approximation ratio 3.72. Following the approach of Jain and Vazirani (2001), Jain et al. (2002) showed that the SCFL can be solved within a factor of 3. This result was further improved by Mahdian et al. (2006) to a 2.89-approximation algorithm for SCFL. Later they present a 2-approximation algorithm for the same problem in Mahdian et al. (2003), achieving the integrality gap of the natural LP relaxation of the problem. This is the best result so far. The main idea of their approach is to consider algorithms and reductions that have separate approximation ratios for facility and connection cost. There is another capacity called hard-capacity in which we cannot open any facility multiple times (cf. Zhang et al. 2002).

In this paper we extend the SCFL and introduce *The 2-Stage Soft-Capacitated Facility Location Problem* (TSSCFL). TSSCFL is defined the same as TSFL except that there is a soft-capacity associated with each facility. By reducing it to *The 2-Stage Linear-Cost Facility Location Problem* (TSLCFL), we are able to approximate TSSCFL within constant factor in polynomial time. The 2-Stage Linear-Cost Facility Location Problem is defined similarly as TSFL except now the cost function is linear on the number of clients each facility serves. More precisely, here $f_i^I = a_i^1 k + b_i^1$ and $f_i^A = a_i^2 k + b_i^2$, where f_i^I is the facility opening cost of the first stage and f_i^A is the facility opening cost of the second stage when scenario A happens. a_i^1 and b_i^1 are constants, called *marginal* and *setup* costs respectively, for the first stage. a_i^2 and b_i^2 are marginal and setup costs respectively, for the second stage. k is the number of clients a facility serves. This problem is also an extension of *The Linear-Cost Facility Location Problem* (cf. Mahdian et al. 2003).

Another way to represent the uncertainty in facility location is to model them as robust problems. *The Robust Fault-Tolerant Facility Location Problem* (RFTFL) is introduced by Chechik and Peleg (2010), who present approximation algorithms of ratio 6.5 and $1.5 + 7.5\alpha$ for the cases of only 1 facility can fail and up to α facilities can fail respectively. In RFTFL, one has to choose a set of facilities that are in a sense robust: i.e., in case of failure of up to α of the opened facilities, where α is viewed as a constant, the cost of connecting clients to the facilities that did not fail should be small. More precisely, we bound the total facility opening cost plus the worst case client connection cost. Byrka et al. (2010a) improve the approximation ratio to $\alpha + 5 + 4/\alpha$. For the case of single failure, Li et al. (2012a) further improve the approximation ratio to 5.236. We show that a better analysis of the algorithm from Byrka et al. (2010a) yields an approximation ratio of $\alpha + 4$.

In this paper we also present a new problem named *The Robust Soft-Capacitated Facility Location Problem* (RSCFL), in which we have to further satisfy the soft-capacity constraint of each facility. *The Robust Linear Cost Facility Location Problem* (RLCFL) is defined analogously as above. Again we use the reduction between them and devise a constant factor approximation algorithm.

The rest of this paper is organized as follows. In Sect. 2 we introduce the formal definitions and reduction between different problems. In Sects. 3 and 4 we provide and analyze approximation algorithms for stochastic and robust version of the problems respectively. We make concluding remarks in Sect. 5. The clients in all problems considered here have only unit demands, except in the dynamic case which will be defined later. It is easy to generalize the unit-demand models to the case of arbitrary demands without loss in approximation guarantee. Each client is connected to only one facility. All the distances satisfy the triangle inequality.

2 Definition

In TSFL, we are given a graph G, a discrete set of clients $\mathcal{D}$, a discrete set of candidate locations $\mathcal{F}$ where to build facilities, two decision stages, and a polynomial number of scenarios each with a prefixed probability p_A for scenario A (A essentially represents a subset of clients). Upon the first stage, we open facilities without the specific knowledge of clients, with facility opening cost f_i^I for each facility i; then in the second stage the realized scenario A enters (a subset of clients appear), we are allowed to open additional facilities but with much higher opening cost f_i^A. Finally connect each client to the nearest opened facility. The aim is to minimize the facility opening cost of the first stage plus the expected cost incurred in the second stage.

TSLCFL is an extension of TSFL, where the facilities opened in two stages incur linear costs. The first stage facility opening cost is $f_i^I = a_i^I k + b_i^I$ for each facility i. a_i^I and b_i^I are the constant marginal cost and setup cost of the first stage, respectively, with $a_i^I \le a_i^A$, and k the number of clients facility i serves. The second stage facility opening cost $f_i^A = a_i^A k + b_i^A$, where a_i^A and b_i^A are the constant marginal cost and setup cost of the second stage for scenario A, respectively.

TSSCFL is an extension of TSFL, where there is a soft-capacity u_i associated with each facility i. If a facility is to serve k clients, we have to open it $\lceil k/u_i \rceil$ times at a cost of $\lceil k/u_i \rceil f_i$.

In RFTFL, we are given a graph G, two discrete sets of clients $\mathcal{D}$, a discrete set of candidate locations $\mathcal{F}$ where to build facilities, and an adversary who can close up to α facilities after opening and assignment. Any client whose serving facility is closed by adversary has to be reassigned to the nearest facility that is still open. The aim is to minimize the sum of the facility opening cost, the connection cost, and the increment of the reassignment cost of the worst case.

The Robust Linear-Cost Facility Location Problem (RLCFL) is an extension of RFTFL, where the facility opening cost is $f_i = a_i k + b_i$ for each facility i. a_i and b_i are the constant marginal cost and setup cost of the first stage, respectively, and k the number of clients facility i serves.

The Robust Soft-Capacitated Facility Location Problem (RSCFL) is an extension of RFTFL, where there is a soft-capacity u_i associated with each facility i. If a facility is to serve k clients, we have to open it $\lceil k/u_i \rceil$ times at a cost of $\lceil k/u_i \rceil f_i$.

3 Reduction

We now build the reductions between the facility location problems mentioned above, based on the reductions developed in Mahdian et al. (2006). We define a variant of TSFL as *The 2-Stage Stochastic Facility Location Problem With Different Connection Cost* (TSFLD), where the service cost of client j connected to facility i is c_{ij}^1 if i is opened in the first stage and is c_{ij}^2 if i is opened in the second stage, with $c_{ij}^1 \le c_{ij}^2$. Here all the c_{ij}^1 and c_{ij}^2 satisfy the triangle inequality respectively.

Lemma 1 *We have a ρ-approximation algorithm for TSLCFL, if there is ρ-approximation algorithm for TSFLD.*

Proof For any instance $\mathcal{I}$ of TSLCFL, we transform it to a new instance $\mathcal{I}'$ of TSFLD with the first stage facility cost $f_i^I = b_i^I$, the second stage facility cost $f_i^A = b_i^A$, and the connection cost $c_{ij}^1 = a_i^I + c_{ij}$ when client j is connected to facility i opened in the first stage, $c_{ij}^2 = a_i^A + c_{ij}$ the second stage. So we have $c_{ij}^1 \le c_{ij}^2$ provided that $a_i^I \le a_i^A$. And it is easy to show that the transformed instance satisfy the triangle inequality.

Lemma 2 *We have a 2ρ-approximation algorithm for TSSCFL, if there is ρ-approximation algorithm for TSLCFL.*

Proof For any instance $\mathcal{I}$ of TSSCFL, we construct an instance $\mathcal{I}'$ of TSLCFL. The connection cost remain the same. The facility cost of facility i is changed to $(1+\frac{k-1}{u_i})f_i$ and $(1+\frac{k-1}{u_i})f_i^A$ if $k \ge 1$ and 0 if $k = 0$. Thus for every $k \ge 1$, $\lceil \frac{k}{u_i} \rceil \le 1 + \frac{k-1}{u_i} \le 2 \cdot \lceil \frac{k}{u_i} \rceil$.

With analogous analysis, we can make similar arguments in the robust case.

Lemma 3 *We have a ρ-approximation algorithm for RLCFL, if there is ρ-approximation algorithm for RFTFL.*

Lemma 4 *We have a 2ρ-approximation algorithm for RSCFL, if there is ρ-approximation algorithm for RLCFL.*

4 The 2-stage soft-capacitated facility location problem

We first show that TSFLD is in fact a special case of DFL, thus it can be solve by the currently best 1.86-approximation algorithm for DFL in Ye and Zhang (2006). DFL is introduced by Roy and Erlenkotter (2002). In the DFL, we are given a set of facilities $\mathcal{F}$, a set of clients $\mathcal{D}$, and the time periods numbered from 1 to T. At each time period t, a client $j \in D$ is specified by a demand d_j^t that can be served by facilities opened at time t or earlier. A cost f_i^t is incurred when the facility $i \in \mathcal{F}$ needs to be open at time period t. A cost c_{ij}^{st} is incurred for supplying one unit demand of client j in time period t from facility i opened at the beginning of time period s ($c_{ij}^{st} = \infty$ for $t < s$). The objective is to choose 0a subset of facilities $\mathcal{F}$ to open at each time period, such that all demands of the clients are satisfied and the total cost is minimized. Here we also assume the triangle inequality: $c_{i_1 j_1}^{st} \le c_{i_2 j_1}^{s't} + c_{i_2 j_2}^{s't'} + c_{i_1 j_2}^{st'}$ for any $i_1, i_2 \in \mathcal{F}$, $j_1, j_2 \in \mathcal{D}$ and time periods $1 \le s \le s' \le t, t' \le T$. It is clearly that if $T = 1$, DFL reduces to UFL.

We present the program for TSFLD. Let $x_{A,ij}^1 = min\{x_{A,ij}, y_i\}$ and $x_{A,ij}^2 = x_{A,ij} - x_{A,ij}^1$. c_{ij} is the cost of connecting client j to i. y_i and $y_{A,i}$ are the extents to which facility i is opened in the first stage and in the second stage under scenario A, respectively. $x_{A,ij}$ is the extent to which j is connected to i in scenario A. We have the following program.

$$
\begin{aligned}
Minimize \sum_{i\in\mathcal{F}} f_i^I y_i + \sum_A p_A \left(\sum_i f_i^A y_{A,i} \right. &+ \left. \sum_{j\in A}\sum_i \left(c_{ij}^1 x_{A,ij}^1 + c_{ij}^2 x_{A,ij}^2 \right) \right) \\
s.\,t. \sum_i x_{A,ij} \geq 1 & \qquad \forall A\ \forall j \in A; \\
x_{A,ij} \leq y_i + y_{A,i} & \qquad \forall i\ \forall A\ \forall j \in A; \\
x_{A,ij}, y_i, y_{A,i} \geq 0 & \qquad \forall i\ \forall A\ \forall j \in A.
\end{aligned}
$$

We now reduce TSFLD to DFL. Suppose the scenario set is $\{A_1, A_2, \ldots, A_m\}$, where m is a fixed integer, and The probability associated with A_k is p_k. Then we set the time periods as $0, 1, \ldots, m$. Set the client set as $\{(k, j), \forall A_k, \forall j \in A\}$. The demand associated with client (k, j) is p_k in period k and is 0 otherwise. The service cost is c_{ij}^1 if client (k, j) is connected to facilities opened in period 0, is c_{ij}^2 in period k and is infinite otherwise. The facility set remains unchanged. The opening cost in the period 0 is f_i^I, and is $p_k f_i^{A_k}$ in period k.

It remains to show that the triangle inequality $c_{i_1 j_1}^{st} \leq c_{i_2 j_1}^{s't} + c_{i_2 j_2}^{s't'} + c_{i_1 j_2}^{st'}$, $j_1, j_2 \in \mathcal{D}$, $1 \leq s \leq s' \leq t, t' \leq T$, is also satisfied. If $s = 0$, we further suppose $s' = t = t'$, otherwise the right side becomes infinity which the inequality naturally follows. Then inequality becomes $c_{i_1 j_1}^1 \leq c_{i_2 j_1}^2 + c_{i_2 j_2}^2 + c_{i_1 j_2}^1$. By the definition of TSFLD, $c_{ij}^1 \leq c_{ij}^2$, for $i \in \mathcal{F}$ and $j \in \mathcal{D}$. Thus the inequality holds. If $s \neq 0$, we can suppose $s = t = s' = t'$, otherwise the both sides of inequality is infinite. In this case the inequality becomes $c_{i_1 j_1}^2 \leq c_{i_2 j_1}^2 + c_{i_2 j_2}^2 + c_{i_1 j_2}^2$ which also follows from the definition of TSFLD.

Thus we have shown that TSFLD is a special case of DFL. We now solve TSFLD with the 1.86-algorithm for DFL and have the following theorem.

Theorem 1 *There is a* 1.86*-approximation algorithm for The 2-Stage Facility Location Problem with Different Connection Cost.*

Considering Lemmas 1 and 2, we have solved TSLCFL and hence TSSCFL.

Theorem 2 *There is a* 1.86*-approximation algorithm for The 2-Stage Linear Cost Facility Location Problem.*

Theorem 3 *There is a* 3.72*-approximation algorithm for TSSCFL.*

5 The robust soft-capacitated facility location problem

We first show that a better analysis of the algorithm for RFTFL from Byrka et al. (2010a) may render an approximation factor of $\alpha + 4$, which improves their result of $\alpha + 5 + 4/\alpha$. The key difference lies in the tighter upper bound for the distance between client and the farthest facility.

The LP relaxation for RFTFL is as follows.

$$
\begin{aligned}
Minimize \sum_{i\in\mathcal{F}} f_i y_i + \max_A \sum_j \sum_i c_{ij} x_{A,ij} & \\
s.t. \sum_i x_{A,ij} \geq 1 & \qquad \forall A\ \forall j; \\
x_{A,ij} \leq y_i & \qquad \forall i\ \forall A\ \forall j; \\
x_{A,ij} = 0 & \qquad \forall A\ \forall i \in A\ \forall j; \\
x_{A,ij}, y_i \geq 0 & \qquad \forall i\ \forall A\ \forall j.
\end{aligned}
$$

Here, $x_{A,ij}$ is the extent to which client j is connected to facility i if the scenario A happens, i.e. the facilities in subset A is closed by the adversary. $|A| \leq \alpha$, where α is the maximal number of facilities the adversary can close. Note that A can be a subset of both opened and not opened facilities.

Let (x^*, y^*) be an optimal fractional solution to the above relaxed LP. We scale it by a factor of γ (which will be set later), i.e. we set $\bar{y}_i = \min\{1, \gamma \cdot y_i^*\}$, $\bar{x}_{A,ij} = \min\{1, \gamma \cdot x^*_{A,ij}\}$. For a single client-scenario pair (j, A), sort the facilities of which $x_{A,ij} > 0$ in nondecreasing order with respect to c_{ij}. Let i' be the first facility in this order such that $x^*_{A,i_1 j} + x^*_{A,i_2 j} + \cdots + x^*_{A,i' j} \geq \frac{\alpha+1}{\gamma}$. We split facilities when necessary to ensure that $x^*_{A,i_1 j} + x^*_{A,i_2 j} + \cdots + x^*_{A,i' j} = \frac{\alpha+1}{\gamma}$. Note that $\sum_i x^*_{A,ij} = 1$ for every j and A. We call the facility set $\{i_1, i_2, \ldots, i'\}$ the *neighborhood* of client j. Then we have the following lemma.

Lemma 5 *Also let $C_{(j,A)} = \sum_i c_{ij} x^*_{A,ij}$ denotes the fractional connection cost of client j in scenario A. Then we have $c_{i'j} \leq \frac{\gamma}{\gamma-\alpha-1} \cdot C_{(j,A)}$, where $c_{i'j}$ is the largest distance between client j and the facilities in the neighborhood of j.*

Proof For a pair (j, A), let $\mathcal{F}' = \{i^1, i^2, \ldots, i'\}$, then $\sum_{i\in\mathcal{F}\setminus\mathcal{F}'} x^*_{A,ij} = 1 - \frac{\alpha+1}{\gamma}$. For $i \in \mathcal{F}\setminus\mathcal{F}'$, we also have $c_{ij} \geq c_{i'j}$. So $C_{(j,A)} = \sum_i c_{ij} x^*_{A,ij} \geq \sum_{i\in\mathcal{F}\setminus\mathcal{F}'} c_{ij} x^*_{A,ij} \geq c_{i'j}(1 - \frac{\alpha+1}{\gamma})$.

The algorithm opens facility randomly as follows.

- If there exist i among $i^1, i^2, \ldots, i'$ such that $\bar{y}_i = 1$. Then facility i will be opened by probability 1. In this case $c_{ij} \leq c_{i'j} \leq \frac{\gamma}{\gamma-\alpha-1} \cdot C_{(j,A)}$.
- If there is no i among $i^1, i^2, \ldots, i'$ such that $\bar{y}_i = 1$, then we round the facilities $i^1, i^2, \ldots, i'$ using the rounding scheme for *The Fault Tolerant Facility Location Problem* (cf. Byrka et al. 2010b), which turns $\alpha + 1$ fractional connections to facilities at distance at most d into $\alpha + 1$ integral connections to facilities at distance at most $3d$. Since at most α facilities will be closed by the adversary, there remains an open facility for client j in scenario A at distance at most $3 \cdot \frac{\gamma}{\gamma-\alpha-1} \cdot C_{(j,A)}$.

With the analysis above, it is clear that we have a $(\gamma, \frac{3\gamma}{\gamma-\alpha-1})$-approximation algorithm for RFTFL. Setting $\gamma = \alpha + 4$, the approximation ratio is $\alpha + 4$.

Theorem 4 *There is a* $(\alpha+4)$*-approximation algorithm for The Robust Fault Tolerant Facility Location Problem, of which up to* α *facilities can be closed by the adversary.*

With Lemma 3, 4, we have the following theorems.

Theorem 5 *There is a* $(\alpha + 4)$*-approximation algorithm for The Robust Linear-Cost Facility Location Problem, of which up to* α *facilities can be closed by the adversary.*

Theorem 6 *There is a* $2(\alpha + 4)$*-approximation algorithm for The Robust Soft-Capacitated Facility Location Problem, of which up to* α *facilities can be closed by the adversary.*

6 Concluding remark

In this paper we have proposed and analyzed the stochastic and robust versions of the soft-capacitated facility location problems. There are relatively very few efforts devoted to designing approximation algorithms to solve combinatorial problems with uncertainty either stochastically or robustly. Incorporating uncertainty into combinatorial problems is a vital way to make these classic models more practical. Approximation algorithms may also become more influential in areas like engineering or operations research if they are used to solve more realistic problems and have good performances. We believe this trend will become more prominent in the future.

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